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Effective Lagrangian for low-lying states of interacting superstrings

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Abstract. The procedure for getting an effective Lagrangian and Hamiltonian for the low-lying states of an interacting string, by integrating over some ultraviolet degrees of freedom, is extended to the fermionic variables. The perturbation on the fermionic levels is determined and a crossed term which describes the mutual effect of the bosonic and fermionic excitations is also obtained.

1. Introduction and summary of the procedure

The derivation of an effective string action, which describes the dynamics of the lower states taking into account the effect of higher excitation, has been considered in many papers, with particular attention to the dynamics of the massless states [1]. A version of this treatment has been presented where the kind of low-lying states is not specified at the beginning, provided that the higher excitations, whose virtual effect must be described by the effective action, are really higher than the states which are explicitly studied. Since this treatment has been presented for the open bosonic string [2] it is natural to extend the same procedure to the fermionic degrees of the open string. The general attitude, as well as the formalism and the notation will be very similar to the one used in [2], which will, hereafter, be referred to as 'the bosonic case'.

The way in which the fermionic degrees of freedom are introduced in the string is not uniquely defined; the model chosen for the present study is the open superstring (with Ramond boundary condition [3]), as is presented, for instance, in the review by Green [4]. All the treatment is done in the light-cone gauge and by dealing explicitly with the partition function; at the end an effective Hamiltonian is derived. In this final derivation the simultaneous presence of bosonic and fermionic variables, which until that moment can be treated separately, has an explicit role. As in [2] only planar configurations will be considered; however, some consideration about the non-planar configurations will be presented at the end.

The interaction of the strings is always described in terms of splitting and rejoining processes, also for the fermionic coordinates [5]; moreover, since the action is quadratic in the chosen gauge, the partition function factorises in the bosonic and fermionic part, and only this last factor will be elaborated in the present paper.

The fermionic factor of the partition is

$$\mathbb{Z}_{\mathrm{f}} = \int_{C_0(0,\beta)} \mathscr{D}\psi\mathscr{D}\psi^{\dagger}\, \mathrm{e}^{-\mathscr{A}_{\mathrm{f}}} + \lambda \int_{C_1(0,\beta;\sigma_0,\tau_0;b)} \mathscr{D}\psi\mathscr{D}\psi^{\dagger}\, \mathrm{e}^{-\mathscr{A}_{\mathrm{f}}} + \ldots \tag{1.1}$$

The second term corresponds to a splitting taking place at σ_0 and lasting from $\tau' = \tau_0 - \frac{1}{2}b$ to $\tau'' = \tau_0 + \frac{1}{2}b$.

The spinorial part of the action is, in the Euclidean world sheet,

$$\mathscr{A}_{\mathrm{f}} = \frac{1}{2\pi} \int_0^\beta \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\, \psi^{\dagger r}(\mathrm{i}\partial_\tau + \boldsymbol{\tau}_z \partial_\sigma)\psi^r \qquad r = 1, \ldots, 4 \tag{1.2}$$

and has been obtained, with the standard position $\psi^r = S^r + \mathrm{i}S^{r+4}$, out of the original expression [4] in terms of the Majorana spinors S,

$$\mathscr{A}_{\mathrm{f}} = \frac{1}{2\pi} \int_0^\beta \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\, \tilde{S}^r(\mathrm{i}\partial_\tau + \boldsymbol{\tau}_z \partial_\sigma) S^r \qquad r = 1, \ldots, 8.$$

The 'external' spinorial index r will be omitted whenever possible; the internal spinorial index takes two values (U, D) and the Pauli matrix $\boldsymbol{\tau}_z$ acts on them†. Following the procedure already described in [2], the second addendum in (1.1) will be recast in the form

$$\int \mathrm{d}\tau_0\, \mathrm{d}\sigma_0 \int_{C_1} \mathscr{D}\psi \mathscr{D}\psi^\dagger\, \mathrm{e}^{-\mathscr{A}_{\mathrm{f}}}$$

$$= \int_{C_0} \mathrm{e}^{-\mathscr{A}_{\mathrm{f}}} \int \mathrm{d}\tau_0\, \mathrm{d}\sigma_0\, \mathscr{E}[\psi(\tau'), \psi(\tau''), \sigma_0] \mathscr{D}\psi \mathscr{D}\psi^\dagger. \tag{1.3}$$

At this point a local leading term will be extracted, which we will call $E(\tau_0, \sigma_0)$. The same considerations already used in the bosonic case allow us to look for an exponentiation of this term, expressing the presence of many non-interacting splitting processes. In this way an effective Lagrangian is obtained:

$$L_{\mathrm{eff}} = L_{\mathrm{T}} - \lambda F - \lambda E \tag{1.4}$$

where L_{T} is the total free Lagrangian, E the effective interaction term of fermionic origin and F the already calculated effective interaction term of bosonic origin.

2. Determination of the interaction term

2.1. Detailed description of the splitting and rejoining process

In this section the actual calculation of the factor $\mathscr{E}$ of equation (1.3) is performed. The elementary process is the fact that the spinorial field $\psi(\sigma, \tau)$ at a certain point σ_0, at an instant τ' splits into two spinorial fields φ and χ, which, in turn, rejoin again into a unique field ψ at the instant τ''. Since the relevant fact of this process is a sudden change of the boundary conditions it is worthwhile starting with a detailed consideration of these conditions and of their conceptual and technical consequences.

The boundary conditions [3] for the spinor ψ, i.e. $\psi_U(0) = \psi_D(0)$, $\psi_U(\pi) = \psi_D(\pi)$, suggest the representation

$$\psi(\sigma) = \sum_l \begin{Vmatrix} \lambda_l \\ \lambda_{-l} \end{Vmatrix} \mathrm{e}^{\mathrm{i}l\sigma} = \sum_l \lambda_l \begin{Vmatrix} \mathrm{e}^{\mathrm{i}l\sigma} \\ \mathrm{e}^{-\mathrm{i}l\sigma} \end{Vmatrix}. \tag{2.1a}$$

† The Pauli matrices are always written in boldface, $\boldsymbol{\tau}_i$, in order to distinguish them from the evolution parameter τ.

For functions of this kind the δ function, in the variable σ, is provided by the matrix

$$\hat{\delta}(\sigma, \sigma') = (2\pi)^{-1} \sum_k \mathrm{e}^{\mathrm{i}k\sigma}(\mathrm{e}^{-\mathrm{i}k\sigma'} + \tau_x \, \mathrm{e}^{\mathrm{i}k\sigma'}) \qquad k = 0, \pm 1, \pm 2, \ldots \tag{2.2}$$

At the value τ' of the evolution parameter τ two spinorial functions are produced, $\varphi(\sigma), \chi(\sigma)$.

They have such boundary conditions, at the point $(0, \sigma_0)$ and (σ_0, π), that they allow the representation

$$\varphi(\sigma) = \sum_l \left\| \begin{matrix} \mu_l \\ \mu_{-l} \end{matrix} \right\| \mathrm{e}^{\mathrm{i}l\pi\sigma/\sigma_0} \tag{2.1b}$$

$$\rho_0 = \pi - \sigma_0.$$

$$\chi(\sigma) = \sum_l \left\| \begin{matrix} \nu_l \\ \nu_{-l} \end{matrix} \right\| \mathrm{e}^{\mathrm{i}l\pi(\pi-\sigma)/\rho_0} \tag{2.1c}$$

When considering functions with these boundary conditions, the representation of the $\hat{\delta}$ function is modified in an obvious way. This modified expression can be used to express the statement that φ (and χ) represent an evolution of ψ which is continuous in τ also at $\tau = \tau'$

$$\left\| \begin{matrix} \mu_l \\ \mu_{-l} \end{matrix} \right\| = \frac{1}{2\sigma_0} \int_0^{\sigma_0} (\mathrm{e}^{-\mathrm{i}l\pi\sigma/\sigma_0} + \tau_x \, \mathrm{e}^{\mathrm{i}l\pi\sigma/\sigma_0}) \psi(\sigma) \, \mathrm{d}\sigma$$

and analogously for ν.

The representation (2.1b) and (2.1c) implies $\varphi_U(\sigma_0) = \varphi_D(\sigma_0)$; $\chi_U(\sigma_0) = \chi_D(\sigma_0)$, which is not, in general, true for $\psi(\sigma_0)$; thus there must be a discontinuity at the point σ_0 both in φ and in χ and therefore a δ singularity in φ' (and χ'), if the evolution parameter is either τ' or τ''. Using the notation already introduced in [2] this singularity can be expressed as

$$\begin{aligned} &\varphi'(\sigma_0) = \lim_{\sigma \to \sigma_0} \varphi'(\sigma) \\ &\partial_\sigma \varphi(\sigma) = \varphi'(\sigma) - \delta(\sigma - \sigma_0)\psi(\sigma_0) \qquad \sigma \leqslant \sigma_0 \end{aligned} \tag{2.3a}$$

and

$$\begin{aligned} &\chi'(\sigma_0) = \lim_{\sigma \to \sigma_0} \chi'(\sigma) \\ &\partial_\sigma \chi(\sigma) = \chi'(\sigma) + \delta(\sigma - \sigma_0)\psi(\sigma_0) \qquad \sigma \geqslant \sigma_0. \end{aligned} \tag{2.3b}$$

There is an obvious difference with respect to the bosonic case: the singularity already shows up at the level of the first derivatives, and this happens because the spinorial part of the original action contains only first derivatives. The general procedure, however, continues to be the same as in the bosonic case, so the interaction term can be obtained starting from the formal quotient

$$\mathscr{E} = \left(\int_{C(\varphi)} \mathscr{D}\varphi \mathscr{D}\varphi^\dagger \, \mathrm{e}^{-\mathscr{A}} \int_{C(\chi)} \mathscr{D}\chi \mathscr{D}\chi^\dagger \, \mathrm{e}^{-\mathscr{A}} \right) \left(\int_{C(\psi)} \mathscr{D}\psi \mathscr{D}\psi^\dagger \, \mathrm{e}^{-\mathscr{A}} \right)^{-1}. \tag{2.4}$$

The relevant τ integration runs from τ' to τ''. Using the standard parametrisation $\tau = \tau' + bu$, $b = \tau'' - \tau'$ the action $\mathscr{A}_\psi$ is

$$\mathscr{A}_\psi = \frac{1}{2\pi} \int_0^1 \mathrm{d}u \int_0^\pi \mathrm{d}\sigma \, \psi^\dagger (\mathrm{i}\partial_u + b\tau_z \partial_\sigma) \psi = \frac{1}{2\pi} \int_0^\pi \mathscr{M}_\psi \, \mathrm{d}\sigma. \tag{2.5}$$

We must now expand ψ with fixed boundary conditions in u:

$$\psi(u) = (1-u)\psi_\mathrm{i} + u\psi_\mathrm{f} + \sum_m \Psi_m f_m(u) \qquad m = 0, \pm 1, \pm 2, \ldots$$

and the third addendum must be zero at the limits $u=0,1$. These conditions cannot be implemented by basis functions f_m, if we ask them to be eigenfunctions of the differential operator, because $\mathrm{i}\partial_u$ is of first order. So the basis f_m is simply chosen to be†

$$f_m = \mathrm{e}^{2\pi \mathrm{i} m u} \qquad m=0, \pm 1, \pm 2, \ldots \tag{2.6}$$

and the constraint shifted to the functional variables Ψ requiring that

$$\sum_m \Psi_m = 0. \tag{2.7}$$

With this kind of expansion, by performing some partial integration in σ (see equation (2.5)) and carrying out the integration in u, the following expression for $\mathcal{M}_\psi$ is obtained:

$$\mathcal{M}_\psi = S_\psi + \sum_n J_n^{(0)\dagger} \Psi_n + \sum_m \Psi_m^\dagger K_m^{(0)} + \sum_{mn} \Psi_m^\dagger D_{mn}^{(0)} \Psi_n \tag{2.8}$$

with the positions‡

$$S_\psi = \mathrm{i}(\psi_\mathrm{f}^\dagger + \psi_\mathrm{i}^\dagger)(\psi_\mathrm{f} - \psi_\mathrm{i}) + \tfrac{1}{3} b[(\psi_\mathrm{f}^\dagger \boldsymbol{\tau}_z \psi_\mathrm{f}' + \psi_\mathrm{i}^\dagger \boldsymbol{\tau}_z \psi_\mathrm{i}) + \tfrac{1}{2}(\psi_\mathrm{f}^\dagger \boldsymbol{\tau}_z \psi_\mathrm{i}' + \psi_\mathrm{i}^\dagger \boldsymbol{\tau}_z \psi_\mathrm{f}')] \tag{2.9a}$$

$$J_n^{(0)+} = \mathrm{i}(\psi_\mathrm{f}^\dagger - \psi_\mathrm{i}^\dagger) + \frac{\mathrm{i}b}{2\pi n}(\psi_\mathrm{f}^{\dagger\prime} - \psi_\mathrm{i}^{\dagger\prime})\boldsymbol{\tau}_z \qquad n \neq 0 \tag{2.9b}$$

$$J_0^{(0)\dagger} = -\tfrac{1}{2} b(\psi_\mathrm{f}^{\dagger\prime} + \psi_\mathrm{i}^{\dagger\prime})\boldsymbol{\tau}_z \tag{2.9c}$$

$$K_n^{(0)} = \frac{\mathrm{i}b}{2\pi n} \boldsymbol{\tau}_z(\psi_\mathrm{f}' - \psi_\mathrm{i}') \qquad n \neq 0 \tag{2.9d}$$

$$K_0^{(0)} = \tfrac{1}{2} b \boldsymbol{\tau}_z(\psi_\mathrm{f}' + \psi_\mathrm{i}') + \mathrm{i}(\psi_\mathrm{f} - \psi_\mathrm{i}) \tag{2.9e}$$

$$D_{mn}^{(0)} = \delta_{mn}(-2\pi n + b\boldsymbol{\tau}_z \partial_\sigma). \tag{2.9f}$$

By introducing explicitly the constraint (2.7) into (2.8) another form is obtained for $\mathcal{M}_\psi$, namely

$$\mathcal{M}_\psi = S_\psi + \sum_n J_n^\dagger \Psi_n + \sum_m \Psi_m^\dagger K_m + \sum_{mn} \Psi_m^\dagger D_{mn} \Psi_n \qquad m, n = \pm 1, \pm 2, \ldots \tag{2.10}$$

with

$$J_n^\dagger = \mathrm{i}(\psi_\mathrm{f}^\dagger - \psi_\mathrm{i}^\dagger) + \tfrac{1}{2} b(\psi_\mathrm{f}^{\dagger\prime} + \psi_\mathrm{i}^{\dagger\prime})\boldsymbol{\tau}_z + \frac{\mathrm{i}b}{2\pi n}(\psi_\mathrm{f}^{\dagger\prime} - \psi_\mathrm{i}^{\dagger\prime})\boldsymbol{\tau}_z \tag{2.11a}$$

$$K_m = -\mathrm{i}(\psi_\mathrm{f} - \psi_\mathrm{i}) - \tfrac{1}{2} b\boldsymbol{\tau}_z(\psi_\mathrm{f}' + \psi_\mathrm{i}') + \frac{\mathrm{i}b}{2\pi m}\boldsymbol{\tau}_z(\psi_\mathrm{f}' - \psi_\mathrm{i}') \tag{2.11b}$$

$$D_{mn} = \delta_{mn}(-2\pi n + b\boldsymbol{\tau}_z \partial_\sigma) + b\boldsymbol{\tau}_z \partial_\sigma. \tag{2.11c}$$

It must be noted that after these transformations the differential operator D is no longer diagonal in (m, n).

In order to perform the functional integration over the anticommuting variables Ψ it is necessary to invert the operator $D_{mn}(\partial_\sigma)$; more precisely it is necessary to determine the matrix Γ such that

$$D_{jm}(\partial_\sigma)\Gamma_{mn}(\sigma, \sigma') = \delta_{jn}\hat{\delta}(\sigma, \sigma'). \tag{2.12}$$

† Usually the discrete variables conjugate to σ are called k, l; those conjugate to u are called m, n, j.

‡ The 'prime' indicates the derivative with respect to σ.

Given the actual expression of $\hat{\delta}$ (2.2) it is natural to look for the following form for Γ:

$$\Gamma_{mn}(\sigma, \sigma') = (2\pi)^{-1} \sum_k \gamma_{mn}(k)\, e^{ik\sigma}(e^{-ik\sigma'} + \tau_x\, e^{ik\sigma'}) \tag{2.13}$$

and it is directly verified that (2.12) is satisfied when one has

$$\gamma_{mn}(k) = \frac{2i\tau_z \tanh(\frac{1}{2}bk)}{(ibk\tau_z - 2\pi m)(ibk\tau_z - 2\pi n)} - \delta_{mn}\frac{1}{ibk\tau_z - 2\pi n} \qquad m, n \neq 0.$$

The fact that γ_{mn} is not diagonal is a direct consequence of the structure of $D(\partial_\sigma)$ in (j, m).

2.2. Functional integration over the fermionic variables

With these tools one can now proceed in performing the functional integration

$$\int \mathscr{D}\Psi\mathscr{D}\Psi^+\, e^{-\mathscr{A}_\psi} = e^{-V_\psi}\Delta_\psi$$

where Δ_ψ is the determinant factor, coming out of the quadratic part of the action and

$$V_\psi = \frac{1}{2\pi}\left(\int d\sigma S_\psi - \sum_{mn}\int d\sigma\, d\sigma'\, J^+_m(\sigma)\Gamma_{mn}(\sigma, \sigma')K_n(\sigma')\right). \tag{2.14}$$

It is now evident that in the same way the other functional integrations appearing in (2.4) are performed resulting in the expressions V_φ and V_χ where the role of ψ both in S and in J and K is played in turn by φ and χ, the limits of σ are respectively between 0 and σ_0 and between σ_0 and π. Moreover two other determinant factors, Δ_φ and Δ_χ, are generated.

In inserting the resulting expressions for V into the quotient $\mathscr{E}$ (2.4) the first term we see is

$$B = \exp\left[(2\pi)^{-1}\int d\sigma(S_\psi - S_\varphi - S_\chi)\right].$$

According to the definition (2.9a) and using the continuity in τ it is evident that for every $\sigma \neq \sigma_0$ the term S_ψ is compensated either by S_φ or by S_χ; at the point $\sigma = \sigma_0$, due to (2.3a) and (2.3b) the contribution from S_φ compensates the contribution from S_χ, so in conclusion $B = 1$. In practice we can drop the terms S everywhere.

In the subsequent treatment the same approach will be employed as was used for the bosonic case, i.e. considering the parameter b small and expanding the expressions in powers of it. To begin we expand γ and Γ:

$$\gamma_{mn}(k) \simeq \frac{1}{2\pi n}\delta_{mn} + ibk\tau_z\frac{1}{(2\pi)^2}\left(\frac{1}{mn} + \delta_{mn}\frac{1}{n^2}\right) + O(b^2) \tag{2.15a}$$

$$\Gamma_{mn}(\sigma, \sigma') \simeq \delta_{mn}\frac{1}{n}\hat{\delta}(\sigma, \sigma') + b\tau_z\frac{1}{2\pi}\left(\frac{1}{mn} + \delta_{mn}\frac{1}{n^2}\right)\partial_\sigma\hat{\delta}(\sigma, \sigma') + O(b^2). \tag{2.15b}$$

After having dropped the term S we can write

$$V_\psi - V_\varphi - V_\chi = -\frac{1}{2\pi}\left(J^+ * \Gamma * K\Big|_\psi - J^+ * \Gamma * K\Big|_\varphi - J^+ * \Gamma * K\Big|_\chi\right) \tag{2.16}$$

where the star product in all cases means the summation over the indices m and n, from $-\infty$ to $+\infty$, excluding the zero, integration in σ from 0 to π for the ψ terms, from 0 to σ_0 for the φ terms, and from σ_0 to π for the χ terms.

The terms J_φ, J_χ, and so on, contain two kinds of addenda, because they depend on the σ derivatives of φ and χ; looking at the definitions (2.3) a first term contains φ, φ' and a second one $\delta(\sigma-\sigma_0)$. Let us write

$$J_\varphi^\dagger=(J_\varphi^\dagger)_C-\delta(\sigma-\sigma_0)(J_\varphi^\dagger)_D. \tag{2.17}$$

Owing to the continuity in τ the first part matches with the corresponding $(J_\psi^\dagger)$ term, while the second is new. Inserting the decomposition (2.17) together with the analogous ones into (2.16) and using for Γ the first term of the expansion (2.15b), we see that all the contributions cancel (as happens for the S terms), except for the case when we take the singular part for both J and K, because in this case the φ and χ terms add up. In this situation the expansion (2.15b) would give rise to a δ^2-like term, and this shows that in this case the expansion itself is not allowed and the full expression of Γ must be used.

In conclusion the contribution of the discontinuous part is explicitly written as

$$V_\psi-V_\varphi-V_\chi|_D=2\sum_{mn}J_{D_m}^\dagger(\sigma_0)\Gamma_{mn}(\sigma_0,\sigma_0)K_{D_n}(\sigma_0) \tag{2.18a}$$

$$J_{D_m}^\dagger(\sigma_0)=\tfrac{1}{2}b\left[(\psi_{\rm f}^{\dagger\prime}+\psi_{\rm i}^{\dagger\prime})+\frac{\rm i}{\pi m}(\psi_{\rm f}^{\dagger\prime}-\psi_{\rm i}^{\dagger\prime})\right]_{\sigma_0}\boldsymbol{\tau}_z \tag{2.18b}$$

$$K_{D_m}(\sigma_0)=\tfrac{1}{2}b\boldsymbol{\tau}_z\left[-(\psi_{\rm f}'+\psi_{\rm i}')+\frac{\rm i}{\pi n}(\psi_{\rm f}'-\psi_{\rm i}')\right]_{\sigma_0} \tag{2.18c}$$

$$\Gamma_{mn}(\sigma_0,\sigma_0)=\frac{1}{2\pi}\sum_k\gamma_{mn}(k)(1+\boldsymbol{\tau}_x) \tag{2.18d}$$

and this last expression holds both if we consider the Γ associated with φ, and defined between 0 and σ_0, and if we consider the Γ associated with χ, and defined between σ_0 and π.

It is a purely technical task now to evaluate the three sums $\Sigma_{mn}\gamma_{mn}(k)$, $\Sigma_{mn}m^{-1}\gamma_{mn}(k)$ and $\Sigma_{mn}(mn)^{-1}\gamma_{mn}(k)$. By changing simultaneously in the term γ: $k\to-k$, $m\to-m$, $n\to-n$ it results that the first and the third term are odd in k and therefore they will yield zero in the subsequent sum over k, so the really relevant sum is the second and for it we get

$$\frac{1}{2\pi}\sum_{mn}\frac{1}{m}\gamma_{mn}(k)=\frac{1}{(bk)^2}\left(1-\frac{2}{bk}\tanh\frac{bk}{2}\right)\equiv A_k.$$

In order to extract the leading term in b, the sum over k is treated with the Euler–McLaurin [6, 7] summation formula obtaining

$$\sum_k A_k\simeq\frac{1}{2b}\int_{-\infty}^{+\infty}\left(1-\frac{\tanh x}{x}\right)\frac{dx}{x^2}=\frac{7}{\pi^2}\zeta(3)\frac{1}{b}.$$

The correction to the formula, which can be found limiting the sum (and integration) between K and $-K$, will produce no power terms in b, because all the derivatives of the integrand go to zero at infinity. The expansion is known [7] to be not convergent, and this signals the presence of an essential singularity in b, as was found, in a simpler way however, for the bosonic case.

Since we are interested in the leading terms in b we perform a b expansion also in J and K using

$$\psi_{\rm i,f}=\psi_C\mp\tfrac{1}{2}b\dot\psi_C+\tfrac{1}{8}b^2\ddot\psi_C+\dots\qquad \psi_C=\psi(\tfrac{1}{2}(\tau'+\tau''))$$

with the result

$$V_{\mathrm{T}} = V_{\psi} - V_{\varphi} - V_{\chi}|_{\mathrm{D}} = \mathrm{i}\, 14\pi^{-3}\zeta(3)b^2[\psi_C^{\dagger}(1-\tau_x)\dot{\psi}_C - \dot{\psi}_C^{\dagger}(1-\tau_x)\psi_C] + \mathrm{O}(b^4). \tag{2.19}$$

Contrary to what occurs in the bosonic case the terms in b^4 unavoidably involve terms containing $\ddot{\psi}$, which are therefore more unusual and of less straightforward interpretation†.

2.3. *Calculation of the determinant*

What has not been taken into account explicitly, up to now, is the relevance of the functional determinant, arising from the integration over the oscillating functions.

As has been noted in (2.11*c*) the differential operator, whose determinant must be calculated, is no longer diagonal. The operator has, however, the structure:

$$M_{mn} = F^{(n)}\delta_{mn} + c$$

and it is found that

$$\det M_{mn} = \prod_n F^{(n)}\left(1 + c\sum_m (F^{(m)})^{-1}\right). \tag{2.20}$$

To prove this result one can start from $M(t) = tF + c$ with the indices m, n running from 1 to N. Then $\det M(t) = P(t)$ is a polynomial in t of order N:

$$P(t) = \sum_{1}^{N}{}_s\, (s!)^{-1} t^s P^{(s)}(0).$$

Using the standard rules to calculate the derivative of the determinant one recognises that only the derivatives of order N and $N-1$ may be different from zero, because the others have at least two equal rows and columns. Since the required result is obtained for $t=1$, the expression in (2.20) is easily derived.

We can now calculate explicitly Δ_ψ, taking the limit $N \to \infty$ whenever possible:

$$\Delta_\psi = \prod_l \prod_{n=-N}^{n=N} (\mathrm{i}bl - 2\pi n)\left(1 + \mathrm{i}bl \sum_m (\mathrm{i}bl - 2\pi m)^{-1}\right)$$

$$= \prod_l (-4\pi^2)^N (N!)^2 \frac{\sinh(\frac{1}{2}lb)}{\frac{1}{2}lb}\, \tfrac{1}{2}lb \coth(\tfrac{1}{2}lb)$$

$$\ln \Delta_\psi = \sum_l [2\ln N! + N\ln(-4\pi^2) + \ln\cosh(\tfrac{1}{2}lb)].$$

When we also calculate $\ln \Delta_\varphi$ and $\ln \Delta_\chi$ we must apply a common cutoff on the wavelength. This cutoff transfers into a cutoff in l which is proportional to π for Δ_φ, to σ_0 for Δ_φ, to ρ_0 for Δ_χ. In this way the terms independent of l cancel in the difference of the logarithms, and every dependence on N vanishes. This property in particular expresses the independence of the final result of the normalisation of the basis functions, (2.6).

† Quantum theories with higher (time) derivatives have, of course, been treated in different contexts [8].

The rest of the procedure is strictly analogous to what was done in [2]: the term $\ln\cosh(\frac{1}{2}lb)$ is expanded in series†

$$\ln\cosh(\tfrac{1}{2}lb)=\sum_r C_r(lb)^{2r}$$

when we consider Δ_ψ, while we get

$$\sum_r C_r(lb\pi/\sigma_0)^{2r} \qquad \text{or} \qquad \sum_r C_r(lb\pi/\rho_0)^{2r}$$

for Δ_φ or Δ_χ.

Now a common regularising procedure is set up

$$\sum_l l^{2r}\to\sum_l l^{2r}\,\mathrm{e}^{-\mu l/\alpha}=(\alpha\partial/\partial\mu)^{2r}[\tfrac{1}{2}\tanh(\mu/2\alpha)-\tfrac{1}{2}]$$

with $\alpha=\pi,\sigma_0,\rho_0$ respectively. Expanding tanh now in μ we verify that only the singular contribution survives in the limit $\mu\to 0$, which yields

$$(2r)!(\alpha/\mu)^{2r+1}.$$

This must be multiplied by $(b\pi/\alpha)^{2r}$ so that the result is

$$(2r)!(b\pi)^{2r}\mu^{-2r-1}(\pi-\sigma_0-\rho_0)=0$$

identically in r; then the artificial cutoff, which however played an important role, can be finally removed and the final result is

$$\ln\Delta_\psi-\ln\Delta_\varphi-\ln\Delta_\chi=0.$$

3. Effective Lagrangian and Hamiltonian

If we limit ourselves to the terms of order b^2, for the fermionic part, then we can proceed rapidly with the next steps. According to the discussion presented in [2] the term obtained can be exponentiated. In this way we get a term $\exp(\lambda\int \mathrm{d}\sigma_0\,\mathrm{d}\tau_0\,\mathrm{e}^{-V_\mathrm{T}})$. It is useful at this point to remember the analogous treatment for the bosonic part, from where we obtained the corresponding term: $\exp(\lambda\int\mathrm{d}\sigma_0\,\mathrm{d}\tau_0\,\mathrm{e}^{T_2})$.

We put together the exponentials and then expand in b remembering, however, that the term $(V_\mathrm{T})^2$, which is also produced, cannot be taken as a correct one since it is of order b^4 and terms of order b^4 were left out in defining V_T.

So finally we can say that to order b^2 we simply get an effective interaction term of the type $\mathrm{i}\lambda\partial\psi^\dagger(1-\boldsymbol{\tau}_x)\psi$; the value of a is obtained from (2.19).

Up until now we have worked with a path integral formalism and with a Euclidean metric, in performing back the time rotation and in setting the operator formalism; this new term, which contains $\dot{\psi}$, has the effect of making $\psi^\dagger$ no longer conjugate to ψ; in order to come back to standard conjugate variables it is necessary to introduce a new field ω by means of the definition

$$\psi=(P+Q\boldsymbol{\tau}_x)\omega.$$

It is easily found that the canonical commutation relation

$$\{\omega(\sigma),\omega^\dagger(\sigma')\}=\delta(\sigma,\sigma')$$

† Using $\ln\cosh x=\int_0^x\tanh y\,\mathrm{d}y$ the coefficients C_r can be expressed in terms of the Bernoulli numbers.

requires

$$(P+Q)^2=1 \qquad (P-Q)^2(1+2a)=1.$$

In this way the original Hamiltonian is transformed in the following way:

$$\int \mathrm{d}\sigma\, \psi^{\dagger} \mathrm{i}\boldsymbol{\tau}_z \partial_\sigma \psi = (1+2a) \int \mathrm{d}\sigma\, \omega^{\dagger} \mathrm{i}\boldsymbol{\tau}_z \partial_\sigma \omega. \tag{3.1}$$

This result shows that, just as in the bosonic case, the only effect we obtained at the order b^2 is a variation of the level splitting which is rigid, i.e. independent of the particular level. In particular, since the modification is multiplicative the zero level is not affected by the correction.

It can also be noted that at the order b^4, taking into account the presence of the bosonic term together with the fermionic one we find a cross term $\mathrm{i}c_1 a x'^2 \psi^{\dagger}(1-\boldsymbol{\tau}_x)\dot{\psi}$. With respect to the variables ψ the x' terms may be treated as numbers so that we find an expression having the same form of (3.1). In this case the result is that, if we consider configurations built up both by fermionic and by bosonic excitations, then among the fermionic levels a variable splitting is introduced, i.e. a splitting which depends on the bosonic population of the configuration†.

4. Conclusions and further considerations

The construction of an effective Hamiltonian representing, in the functional space of the free string, the effect of string interaction in a well defined limit has been extended to the fermionic coordinates. The extension has been performed following as strictly as possible the procedures used to deal with the bosonic case. Some technical differences and a certain amount of formal complication turned out to be unavoidable; they have not, however, been very large because the aim of the paper is limited to the sole investigation of the partition function. So the more elaborate formal procedures needed to project the external fermionic states are not required [5, 9]. In so doing, only the static effects of the higher-frequency dynamics can be determined, in the form of level shifting and level splitting, while the effects on the scattering amplitudes, which are certainly present, are not investigated. The static effects are of the same type as for the bosonic case, since the effective Hamiltonian contains also quartic terms: a cross term between bosonic and fermionic coordinates is produced. The calculation of a pure quartic term in the fermionic variables is a purely technical problem although complicated and tedious. The exponentiation procedure, which ultimately allows the definition of the effective action raises two questions. The first is the possibility of representing the effects of splitting and rejoining as a local interaction term. This possibility was already discussed in [2] and reconsidered in the present paper. The definition of terms up to the fourth order in the coordinates leads to leading local expressions. On the contrary, by going further in the powers of coordinates, the aspect of the result changes. To be consistent, in the expansion in b, essentially non-local terms come into the game; at this point the exponentiation becomes more questionable. One could still look for a locality in τ (for b small) but an effective action non-local

† Evidently the statement may be reversed: there is a variable splitting in the bosonic levels, depending on the fermionic population.

in σ does not give an interesting insight into the problem; it is better to say that the whole procedure is consistent and interesting up to the quartic terms in the coordinates, both bosonic and fermionic. The second question concerns the role of more complicated topologies. A preliminary investigation has been performed for the typical case where there is a twist of one of the two fragments of the string. When this happens, between the splitting and rejoining point a non-orientable evolution surface is generated. If this process happens at fixed σ_0 and the limit $b \to 0$ is taken then a singularity is generated in terms like V, or like T in the bosonic case where this investigation was done. The singularity in turn yields a non-local term and it can be foreseen that the overall effect of such a term on the partition function $\mathbb{Z}$ is to produce a term going to zero faster than any power of b. This result is, however, true at fixed σ_0; if this is not the case, and we let σ_0 go to zero (or even to π) together with b, the zero is much milder. So this preliminary look at the problem suggests that the introduction of twisted configurations will give rise to effective terms concentrated at the endpoints of the open string. The general considerations presented in [2] about the relationship of this procedure to other treatments apply also to the present extension. So the whole treatment is a way of representing some leading ultraviolet effects of higher degrees of freedom in terms of a local effective Lagrangian and yields an unambiguous answer—the extension to dynamical problems, i.e. to scattering amplitudes, involves non-trivial difficulties but there is nothing in the procedure that prevents this possibility.

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